

Quantum Stability of the Classically Instable $(-\phi^6)$ Scalar Field Theory

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Abstract

In this work, we show that stability of a theory can not be predicted from classical analysis. Regarding this, we study the stability of the bounded from above $(-\phi^6)$ scalar field theory where classical analysis prohibits the existence of a stable spectrum. We calculated the effective potential up to first order in the couplings in d space-time dimensions. We find that a Hermitian effective theory is instable while a non-Hermitian but \mathcal{PT} -symmetric effective theory characterized by a pure imaginary vacuum condensate is rather stable (bounded from below) which is against the classical predictions of the instability of the theory. This is the first calculations that advocates the stability of the $(-\phi^6)$ scalar potential. Apart from these interesting results, we showed that the effective field approach we followed is able to reproduce the very recent results presented in Physical Review Letters 105, 031601 (2010) for the corresponding bounded from below theory.

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The very active research area of the non-Hermitian theories with real spectra [1–9] may offer solutions to current existing problems in our understanding of nature. Among the very large number of non-Hermitian theories investigated, theories with bounded from above potentials deserve more interest than that offered in the literature. The reason for that is the possessing of the very important asymptotic freedom property in the quantum field versions out of these theories [2, 10–12]. To shed light on the importance of this property one has to mention that in the past, to have such interesting property, physicists had to resort to a somehow complicated theory that merge group theory to field theory with the number of colors to be equal to or greater than three (quantum chromodynamics). Now and after the discovery of possible physical acceptability of non-Hermitian theories, one can get the important asymptotic freedom property from just colorless one component scalar field theories.

The fruitfulness of the discovery of the possible physical acceptability of some of the non-Hermitian theories goes even beyond the existence of the asymptotic freedom property in a simple theory. In Ref.[13], we showed that a new matter phase can exist in which the theory is no longer Hermitian but \mathcal{PT} -symmetric. Moreover, in Ref.[5], a finite non-Hermitian theory has been shown to be equivalent to a divergent theory which gives a hope (in principle) to the killing of divergences existing even in non-renormalizable theories [26]. Besides, in Ref.[3], we showed that the mathematical tools associated with the non-Hermitian theories can be used to cure the ghost states in the Lee-Wick standard model. In fact, the Lee-Wick standard model is known to be free from the famous Hierarchy problem without the need of introducing the so far undiscovered s-particles in the supersymmetric extension of the standard model. In the same line, we have argued in Ref.[2] that a \mathcal{PT} -symmetric Higgs mechanism is strongly believed to kill the Hierarchy puzzle too.

Apart from the above mentioned benefits that can be obtained from the employment of the non-Hermitian theories in our modeling of natural events, a big problem was thought to exist in dealing with such theories. In fact, for physical amplitude calculations in the non-Hermitian theories, the metric operator formulations are indispensable. However, the suggested regimes for metric operator in the literature turns the calculations divergent even at the quantum mechanical case [5]. For Higher dimensions, the degree of divergences will be even higher and the calculation of the metric operator becomes complicated and even hard to get it in a closed form for some perturbative calculations [14]. However, Jones and

Rivers showed that in case of metric operator of gauge form one can get physical amplitudes from path integrals within the non-Hermitian theory [15]. Moreover, it has been shown that the effective field approach knows about the metric [16] and since effective field approach can be easily extended to the important quantum field case [2, 4], one may not worry about the metric any more.

The bounded from above theories do have another problem. The point is that, classically, one can not advocate the stability of such theories and since the quantum world behaves in many situations very strange (like existence of tunneling as a pure quantum effect), one might expect that quantum mechanically such theories are stable. Stability in these theories can be advocated via the numerical calculations of the spectra and if they have a lower ground state then stability is assured. For the bounded from above ($-x^4$) theory, the authors of Ref. [17] followed analytic calculations and obtained an equivalent bounded from below x^4 theory with anomaly. In fact, neither numerical calculations nor the complex contour methods can be extended easily to higher dimensions (quantum field theory). Accordingly, our aim in this work is to study the bounded from above ($-\phi^6$) field theory (for the first time) in d space-time dimensions and show that the vacuum is in fact stable which is against the classical picture. In fact, the corresponding bounded from below (ϕ^6) theory has been studied very recently and has been shown to have a vacuum solution with unbroken Z_2 symmetry and another one with broken Z_2 symmetry. The authors of Ref.[18] attributed these vacua to the existence of two kinds of particles (family). So in our work, we will reproduce their results first as a kind of a test to the effective field calculations we follow and also show some thing new that the theory is stable in the broken symmetry phase too which has not been shown in Ref[18].

To start, consider the quantum field Hamiltonian density of the form;

$$H = \frac{1}{2} (\nabla\phi)^2 + \frac{1}{2}\pi^2 + \frac{1}{2}m^2\phi^2 + g\phi^4 + h\phi^6, \quad (1)$$

where m is the mass of the field ϕ , π is the conjugated momentum field while g and h are coupling constants.

The mean field approach is lunched by the canonical transformation $\phi = \psi + B$ and $\pi = \Pi$. Here, B is a constant called the vacuum condensate and $\Pi = \dot{\psi}$. Plugging these

transformations into the Hamiltonian model in Eq.(1) to get an equivalent effective form as;

$$\begin{aligned}
H = & \frac{1}{2} (\nabla\psi)^2 + \frac{1}{2}\Pi^2 + \frac{1}{2}M^2\psi^2 + \left(15hB^4 + 6gB^2 + \frac{1}{2}m^2 - \frac{1}{2}M^2\right) \psi^2 \\
& + (20hB^3 + 4gB) \psi^3 + (15hB^2 + g) \psi^4 + 6Bh\psi^5 + h\psi^6 \\
& + \left(hB^6 + gB^4 + \frac{1}{2}B^2m^2\right),
\end{aligned} \tag{2}$$

where we dropped out the linear term in the field ψ since one can constrain the parameters to secure the stability of the theory and this term will disappear order by order [19]. Moreover, we have chosen to work with the mass parameter M of the field ψ and consider all terms except the kinetic term $(\frac{1}{2}(\nabla\psi)^2 + \frac{1}{2}\Pi^2 + \frac{1}{2}M^2\psi^2)$ to constitute an interaction Hamiltonian. The vacuum expectation value of the Hamiltonian operator $\langle 0|H|0\rangle$ is known as the effective potential or vacuum energy. Up to first order in the couplings one can get the result for d space-time dimensions as;

$$E = \frac{1}{2(4\pi)^{\frac{d-1}{2}}} \left(\frac{\Gamma(-\frac{1}{2} - \frac{d-1}{2})}{\Gamma(-\frac{1}{2})} \left(\frac{1}{\Omega^2} \right)^{-\frac{1}{2} - \frac{d-1}{2}} \right) + \frac{1}{2}B^2m^2 + gB^4 + hB^6 + \left(\frac{-i6!h}{-3! \times 8i} \right) (\Delta)^3 \tag{3}$$

$$+ \left(\frac{-i4!(15hB^2 + g)}{-i8} \right) (\Delta)^2 + \left(\frac{-i2(15hB^4 + 6gB^2 + \frac{1}{2}m^2 - \frac{1}{2}M^2)}{-i2} \right) (\Delta), \tag{4}$$

where

$$\Delta = \frac{1}{(4\pi)^{\frac{d}{2}}} \frac{\Gamma(1 - \frac{d}{2})}{\Gamma(1)} \left(\frac{1}{M^2} \right)^{1 - \frac{d}{2}}$$

and $E = \langle 0|H|0\rangle$ and Γ is the gamma function. For $d = 1$ (quantum mechanics), E can be simplified as;

$$E = \frac{1}{2}M + \frac{1}{2}B^2m^2 + gB^4 + hB^6 + \frac{3}{4} \frac{(15hB^2 + g)}{M^2} + \frac{15}{8} \frac{h}{M^3} + \frac{1}{4M} (30hB^4 + 12gB^2 - M^2 + m^2).$$

For this effective potential to be stable, one has to constrain the parameters introduced in the calculations such that;

$$\frac{\partial E}{\partial B} = 0.$$

For $g = 0$ and $m = 0$, we get the result;

$$\frac{3}{2} \frac{B}{M^2} h (4B^4M^2 + 20B^2M + 15) = 0. \tag{5}$$

This equation has three different solutions of the form;

$$\begin{aligned} B &= 0, \\ B^2 &= -\frac{1}{2M} \left(\sqrt{10} + 5 \right), \\ B^2 &= \frac{1}{2M} \left(\sqrt{10} - 5 \right). \end{aligned} \tag{6}$$

The $B = 0$ solution is acceptable only for the bounded from below theory (positive h). In this case the theory is Hermitian and the vacuum is stable as well. For the solutions $B^2 = -\frac{1}{2M} (\sqrt{10} + 5)$ and $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$, M is positive and thus B is imaginary. Accordingly, the Hamiltonian form in Eq.(2) is non-Hermitian but \mathcal{PT} -symmetric as well and one then can claim that the spectrum of the theory is real and stable for both broken symmetry solutions in Eq.(6) either h positive or negative. In fact, the story here is different and it is only the solution $B^2 = -\frac{1}{2M} (\sqrt{10} + 5)$ that is stable for the bounded from below potential ($+h$) while the solution $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$ resembles an unstable vacuum. In fact, for the solution $B^2 = -\frac{1}{2M} (\sqrt{10} + 5)$, the effective potential has the form;

$$E = -\frac{1}{B^2 (136\sqrt{10} + 440)} \left((64h + 32\sqrt{10}h) B^8 + 140\sqrt{10} + 445 \right),$$

which shows that E with B real is $-E$ with B imaginary. In fact, real B means that M is negative which means the existence of ghost states (negative kinetic energy). This means that the effective field approach we followed accounts for the metric since with imaginary B the kinetic energy is positive. This understanding agrees well with the predictions of Ref.[16] that mean field approach knows about the metric. Moreover, one can then get the same ground state if we consider the problem as of negative mass particle (Hermitian but Lee-Wick theory) that maximizes the action or a positive mass particle that minimizes the action (non-Hermitian theory).

In this article we shall stick to the usual understanding of particles as they have positive masses and stability exists from minimizing actions. Consequently, B is chosen imaginary and to investigate the stability of the theory we plot the diagram in Figs.1 where one can realize that the effective potential (vacuum energy) is bounded from below for $h = +\frac{1}{2}$ for the solution $B^2 = -\frac{1}{2M} (\sqrt{10} + 5)$. On the other hand, the solution $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$ results in an instable vacuum since the associated effective potential is unbounded either from above or from below (Fig. 2). Again with the solution $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$, the

effective potential has the form;

$$E = \frac{1}{B^2 (136\sqrt{10} - 440)} \left((64h - 32\sqrt{10}h) B^8 - 140\sqrt{10} + 445 \right),$$

which for B real has exactly an opposite sign out of B imaginary result.

The solution $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$, however, results in a stable effective potential for an unstable classical potential (negative coupling) as shown in Fig.3 while the solution $B^2 = -\frac{1}{2M} (\sqrt{10} + 5)$ is unstable (Fig.4). These results are in fact very interesting since they show that stability (like tunneling) can not be argued in view of classical analysis. A classically stable potential may or may not lead to a stable quantized system. The reverse is also correct, a classically instable potential can have stable as well as instable spectra.

To show the compatibility of our calculations with other approaches, let us test it in view of the very recent work in Ref. [18]. For positive h and for $0 + 1$ space-time dimensions the authors found the mass ratio $\frac{M_B}{M_0} = 1.6197$ where the mass of the broken symmetry phase is M_B and M_0 characterizes the symmetric phase. To compare our result to this one, we employ the fact that the effective potential is the generating functional of the one particle irreducible amplitudes [19]. Consequently, one can employ the relation $\frac{\partial^2 E}{\partial B^2} = M^2$. This relation in conjunction with the relation $\frac{\partial E}{\partial B} = 0$, results in the following equations for $B \neq 0$,

$$\begin{aligned} h(4B^4 M^2 + 20B^2 M + 15) &= 0 \\ \frac{15}{2M^2} h(4B^4 M^2 + 12B^2 M + 3) &= M^2, \end{aligned} \tag{7}$$

and the relation $M_0 = \sqrt[4]{\frac{45h}{2}}$ for $B = 0$. The equation set (7) can be solved to give; $M_B = \sqrt[4]{30(\sqrt{10} + 2)h}$ and $B^2 = -\frac{1}{2M_B} (\sqrt{10} + 5)$, which assures our previous assumption that B should be imaginary. Accordingly, one can obtain the ration $\frac{M_B}{M_0} = 1.6197$, which is exactly the same result in Ref.[18] although our approach is different. For the bounded from above potential (negative h) the $B = 0$ solution is unstable. Accordingly, only the broken symmetry solution characterized by the parameters $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$ and $M = \sqrt[4]{-30(\sqrt{10} - 2)h}$ is the only acceptable solution. The behavior of the vacuum condensate as a function of the coupling h is shown in Fig.5 while the behavior of the effective mass is presented in Fig.6.

In d space-time dimensions, for the massive theory as well as for $g \neq 0$, we get the result;

$$M = \frac{1}{\left(15^{-\frac{1}{-2+d}}\right)} 2^{-\frac{d}{-2+d}} \pi^{\frac{d}{2(-2+d)}} \frac{\left(-2g - 10B^2h + \sqrt{4g^2 + 40B^4h^2 - 10hm^2}\right)^{-\frac{1}{-2+d}}}{\left(h^{-\frac{1}{-2+d}}\right) \left((- \Gamma(-\frac{1}{2}d) d)^{-\frac{1}{-2+d}}\right)},$$

$$M = \frac{1}{\left(15^{-\frac{1}{-2+d}}\right)} 2^{-\frac{1}{-2+d}d} \pi^{\frac{1}{2(-2+d)}d} \frac{\left(-2g - 10B^2h - \sqrt{4g^2 + 40B^4h^2 - 10hm^2}\right)^{-\frac{1}{-2+d}}}{\left(h^{-\frac{1}{-2+d}}\right) \left((- \Gamma(-\frac{1}{2}d) d)^{-\frac{1}{-2+d}}\right)} \quad (8)$$

For the $0 + 1$ case, we get;

$$\frac{E}{m} = e = -Hb^6 + \left(\frac{15}{2t}H + G\right)b^4 - \left(\frac{45}{4t^2}H + \frac{3}{t}G + \frac{1}{2}\right)b^2 + \frac{1}{4}t + \frac{3}{4t^2}G + \frac{15}{8t^3}H + \frac{1}{4t}, \quad (9)$$

where we put $B = m^{\frac{d-2}{2}} \frac{b}{i}$, $h = Hm^{-2d+6}$, $g = Gm^{-d+4}$ and $M = tm$. In this case we get the dimensionless mass $t = \frac{M}{m}$ in the form;

$$t = \frac{-15H}{2G - 10b^2H + \sqrt{4G^2 + 40b^4H^2 - 10H}},$$

$$t = \frac{-15H}{2G - 10b^2H - \sqrt{4G^2 + 40b^4H^2 - 10H}}.$$

As we can see from Fig.7 the effective potential is bounded from below (stable) for negative H and for $\pm G$ but only for the solution;

$$t = \frac{-15H}{2G - 10b^2H + \sqrt{4G^2 + 40b^4H^2 - 10H}}.$$

For higher space-time dimensions, the dimensional regularization used to calculate the Feynman diagrams leaded to the above results may not be able to get rid of the existing divergences. For instance, in $1 + 1$ space-time dimensions, the gamma function, $\Gamma(1 - \frac{1}{2}d)$, is divergent and thus one may resort to another regularization tool like minimal subtraction. Another point that one has to care about is the invariance of the bare parameters under the renormalization group. However, as long as we constrain our selves to first order in the couplings, normal ordering can overcome these two problems [2, 20–25]. In the following, we will use the normal ordering technique to study the cases $1 + 1$ and $2 + 1$ while the $3 + 1$ case will be skipped due to the non-renormalizability of the theory.

The normal ordering of the field operators follows the relations [20];

$$\begin{aligned}
N_m \psi &= N_M \psi, \\
N_m \psi^2 &= N_M \psi^2 + \Delta, \\
N_m \psi^3 &= N_M \psi^3 + 3\Delta N_M \psi, \\
N_m \psi^4 &= N_M \psi^4 + 6\Delta N_M \psi^2 + 3\Delta^2, \\
N_m \psi^5 &= N_M \psi^5 + 10\Delta N_M \psi^3 + 15\Delta^2 \psi, \\
N_m \psi^6 &= N_M \psi^6 + 15\Delta N_M \psi^4 + 45\Delta^2 \psi^2 + 15\Delta^3,
\end{aligned}$$

while the normal ordering of the kinetic term gives;

$$N_m \left(\frac{1}{2} (\nabla \psi)^2 + \frac{1}{2} \Pi^2 \right) = N_M \left(\frac{1}{2} (\nabla \psi)^2 + \frac{1}{2} \Pi^2 \right) + E_0(M) - E_0(m), \quad (10)$$

where

$$E_0(\Omega) = \frac{1}{4} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \left(\frac{2k^2 + \Omega^2}{\sqrt{k^2 + \Omega^2}} \right) = I_1 + I_2,$$

with

$$I_1(\Omega) = \frac{1}{2} \frac{1}{(4\pi)^{\frac{d-1}{2}}} \frac{d}{2} \left(\frac{\Gamma\left(\frac{1}{2} - \frac{d-1}{2} - 1\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{\Omega^2}\right)^{\frac{1}{2} - \frac{d-1}{2} - 1} \right), \quad (11)$$

$$I_2(\Omega) = \frac{\Omega^2}{4} \frac{1}{(4\pi)^{\frac{d-1}{2}}} \left(\frac{\Gamma\left(\frac{1}{2} - \frac{d-1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} \left(\frac{1}{\Omega^2}\right)^{\frac{1}{2} - \frac{d-1}{2}} \right). \quad (12)$$

In 1 + 1 dimensions, we get $\Delta = -\frac{1}{4\pi} \ln t$ and

$$E_0(M) - E_0(m) = \frac{m^2}{8\pi} (t - 1 - \ln t)$$

where $t = \frac{M^2}{m^2}$. Accordingly, the vacuum energy can take the form;

$$\begin{aligned}
\frac{8\pi E}{m^2} = e &= -2Hb^6 + (-30H \ln t + G) b^4 - (90H \ln^2 t - 6(\ln t) G + 1) b^2 \\
&+ 3(\ln^2 t) G - 30H \ln^3 t + t - 1 - \ln t,
\end{aligned}$$

which is constrained by the equation $\frac{\partial E}{\partial B} = 0$ or ;

$$2b(6Hb^4 - (2G - 60H \ln t) b^2 + 90H \ln^2 t - 6(\ln t) G + 1) = 0. \quad (13)$$

In the above forms we used the parameterizations $g = 2\pi G m^2$, $B = \frac{b}{i\sqrt{4\pi}}$ and $h = (4\pi)^2 H m^{-2d+6}$. For $b \neq 0$, Eq.(13) has the solutions,

$$\begin{aligned}
t &= \exp \left(\frac{1}{180H} \left(-60Hb^2 + 6G + 6\sqrt{40H^2b^4 + G^2 - 10H} \right) \right), \\
t &= \exp \left(\frac{1}{180H} \left(-60Hb^2 + 6G - 6\sqrt{40H^2b^4 + G^2 - 10H} \right) \right).
\end{aligned} \quad (14)$$

For the bounded from above case $(-h)$, the solution

$$t = \exp \left(\frac{1}{180H} \left(-60Hb^2 + 6G + 6\sqrt{40H^2b^4 + G^2 - 10H} \right) \right),$$

is the stable one (Fig.8).

In $2 + 1$ dimensions, we get $E_0(M) - E_0(m) = \frac{1}{24\pi} (M^3 - m^3)$ and $\Delta = \frac{1}{4\pi} \frac{m-M}{\pi}$. Using the parameterizations $B = m^{\frac{d-2}{2}} \frac{b}{i\sqrt{(4\pi)}}$, $g = (4\pi) G m^{-d+4}$, $h = (4\pi)^2 H m^{-2d+6}$ and $M = tm$ we obtain the results;

$$\begin{aligned} \frac{4\pi E}{m^3} = e = & -Hb^6 + (15H(1-t) + G)b^4 - \left(\frac{1}{2} + 45H(t-1)^2 + 6G(1-t) \right) b^2 \\ & + 3G(t-1)^2 + 15H(1-t)^3 + \frac{1}{6}(t+2)(t-1)^2, \end{aligned}$$

and

$$\begin{aligned} t = & \frac{1}{180H} \left(180H - 60Hb^2 + 12G + 6\sqrt{40H^2b^4 + 4G^2 - 10H} \right), \\ t = & \frac{1}{180H} \left(180H - 60Hb^2 + 12G - 6\sqrt{40H^2b^4 + 4G^2 - 10H} \right). \end{aligned}$$

Only the solution

$$t = \frac{1}{180H} \left(180H - 60Hb^2 + 12G + 6\sqrt{40H^2b^4 + 4G^2 - 10H} \right),$$

results in a stable effective potential for the bounded from above $(-\phi^6)$ potential (Fig.9).

To conclude, we calculated the effective potential of the \mathcal{PT} -symmetric $(-\phi^6)$ theory for a d space-time dimensions. The classical potential of this theory is bounded from above and thus has not been stressed in the literature due to the believe that this theory is unstable. We have shown that as long as the vacuum condensate is imaginary, the effective Hamiltonian is non-Hermitian but \mathcal{PT} -symmetric and the effective potential is rather bounded from below which proves the stability of the theory. In fact, the we get three different vacuum solutions however we figured out that the effective potential is bounded from below for only one vacuum solution out of the three available solutions. The other two vacuum solutions are in fact stabilizing the corresponding bounded from below theory studied very recently in Ref [18]. We were able to reproduce exactly their results and even more were able to advocate the stability of their results. In fact, a great lesson can be learned from this work because it shows that bounded from above potentials can have both stable and unstable vacuum solutions and the bounded from below potentials can have stable as well as unstable

solutions too. Accordingly, one may expect bouncing off as well as formation of bound states when incident particles are scattered from either bounded from below or bounded from above potentials. Predictions of such events in the lab will offer a great support to the believe in the \mathcal{PT} -symmetric theories.

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[26] In another work (submitted), we showed that the divergences found in the Hermitian equivalent theory in Ref.[5] were due to supernormalizable, renormalizable and non-renormalizable terms in the interaction Hamiltonian while the non-Hermitian equivalent theory is even finite.

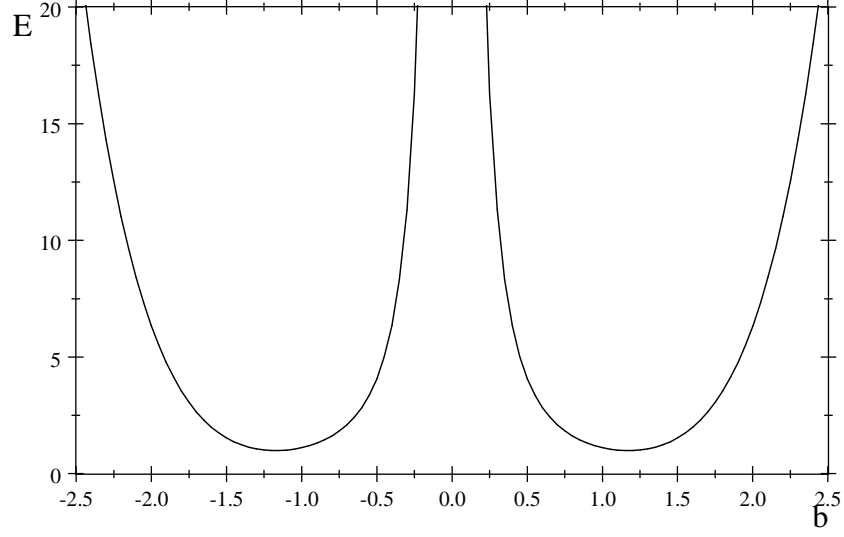


FIG. 1: The ground state energy as a function of the one-point function $B \equiv \langle 0|\phi|0\rangle \equiv ib$ measured in units of i for $h = 0.5$ for the solution $B^2 = -\frac{1}{2M}(\sqrt{10} + 5)$ of the \mathcal{PT} -symmetric ϕ^6 potential.

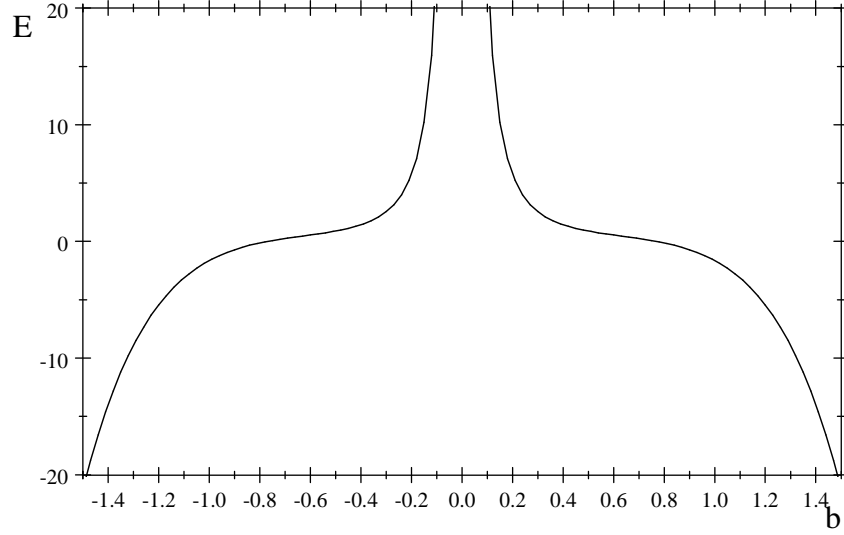


FIG. 2: The ground state energy as a function of the one-point function measured in units of i for $h = 0.5$ for the the solution $B^2 = \frac{1}{2M}(\sqrt{10} - 5)$ of the \mathcal{PT} -symmetric ϕ^6 potential.

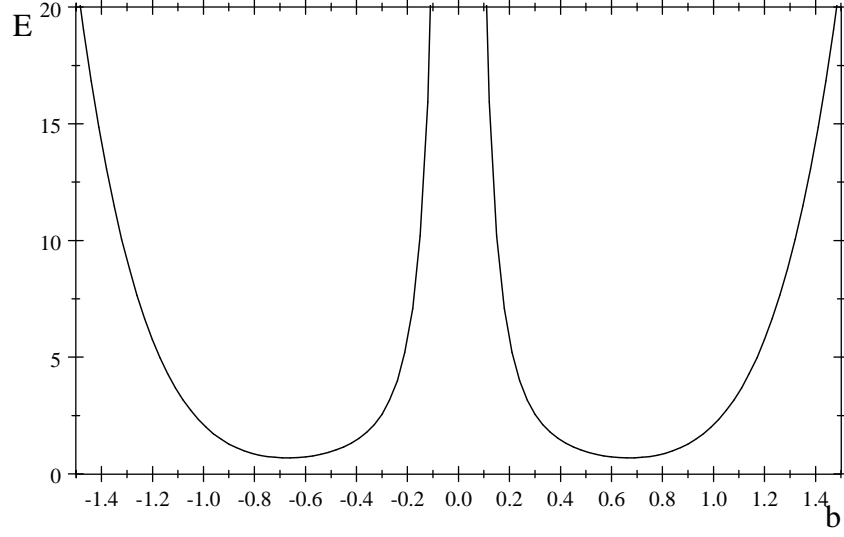


FIG. 3: The ground state energy as a function of the one-point function measured in units of i for $h = -0.5$ for the the solution $B^2 = \frac{1}{2M} (\sqrt{10} - 5)$ of the \mathcal{PT} -symmetric ϕ^6 potential.

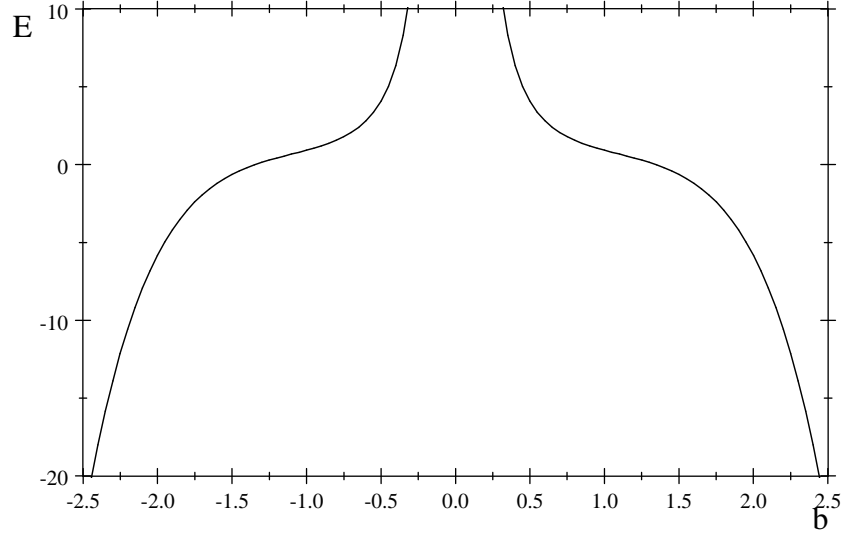


FIG. 4: The ground state energy as a function of the one-point function measured in units of i for $h = -0.5$ for the solution $B^2 = -\frac{1}{2M} (\sqrt{10} + 5)$ of the \mathcal{PT} -symmetric ϕ^6 potential.

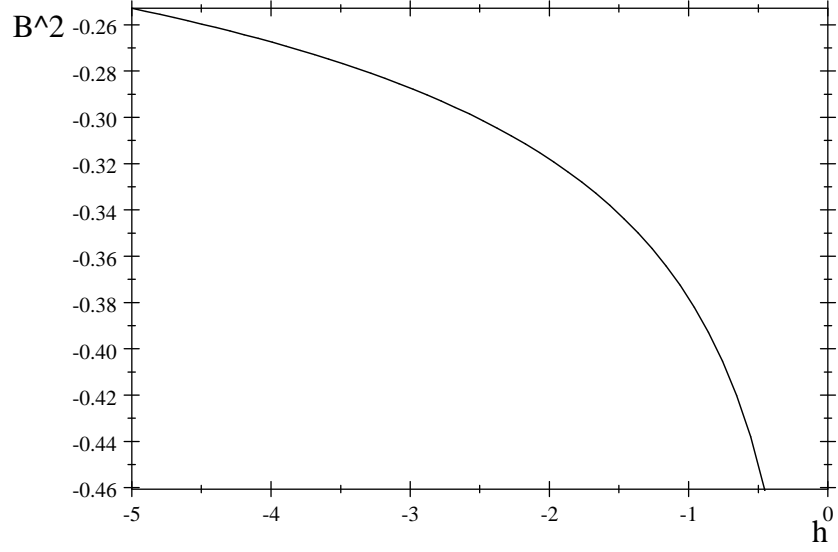


FIG. 5: The one-point function squared versus the coupling h for the massless \mathcal{PT} -symmetric $(-\phi^6)$ potential in $0 + 1$ space-time dimensions.

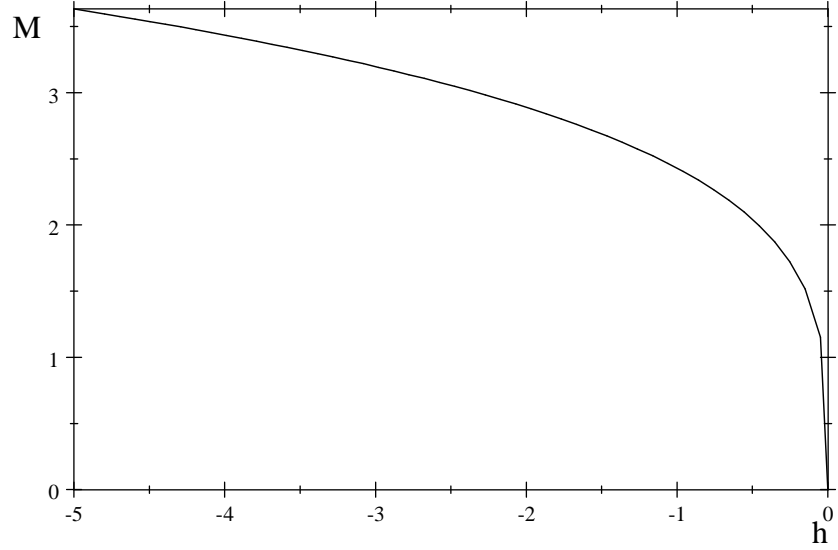


FIG. 6: The effective mass of the field ψ versus the coupling h for the massless \mathcal{PT} -symmetric $(-\phi^6)$ potential in $0 + 1$ space-time dimensions.

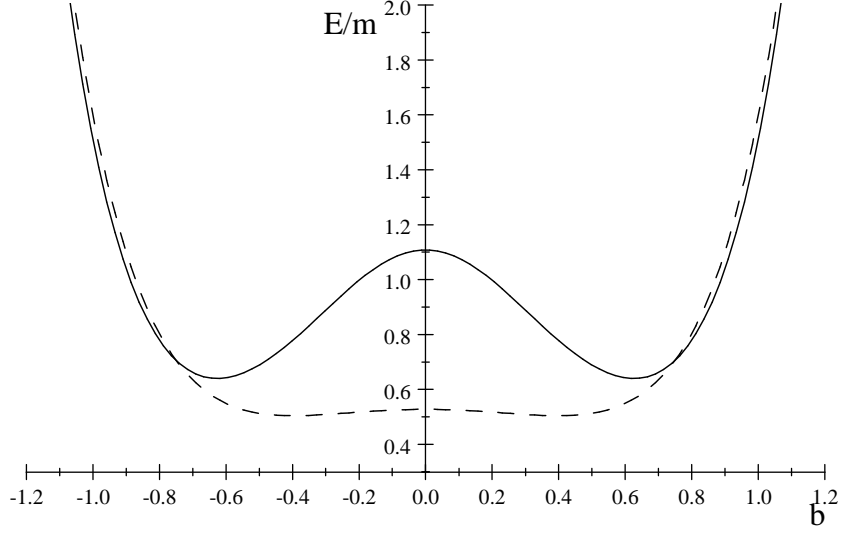


FIG. 7: The 0 + 1 space-time dimensions effective potential as a function of the one-point function measured in units of i for $H = -\frac{1}{4}, G = +\frac{1}{2}$ (dashed) and $H = -\frac{1}{4}, G = -\frac{1}{2}$ (solid) with the stable solution $t = \frac{-15H}{2G-10b^2H+\sqrt{4G^2+40b^4H^2-10H}}$ of the \mathcal{PT} -symmetric ϕ^6 potential.

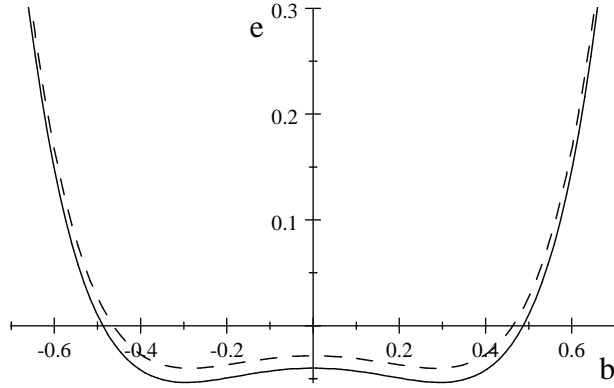


FIG. 8: The 1 + 1 space-time dimensions effective potential as a function of the one-point function measured in units of i for $H = -\frac{1}{4}, G = +\frac{1}{2}$ (dashed) and $H = -\frac{1}{2}, G = -\frac{1}{2}$ (solid) with the stable solution $t = \exp\left(\frac{1}{180H}\left(-60Hb^2 + 6G + 6\sqrt{40H^2b^4 + G^2 - 10H}\right)\right)$ of the \mathcal{PT} -symmetric ϕ^6 potential .

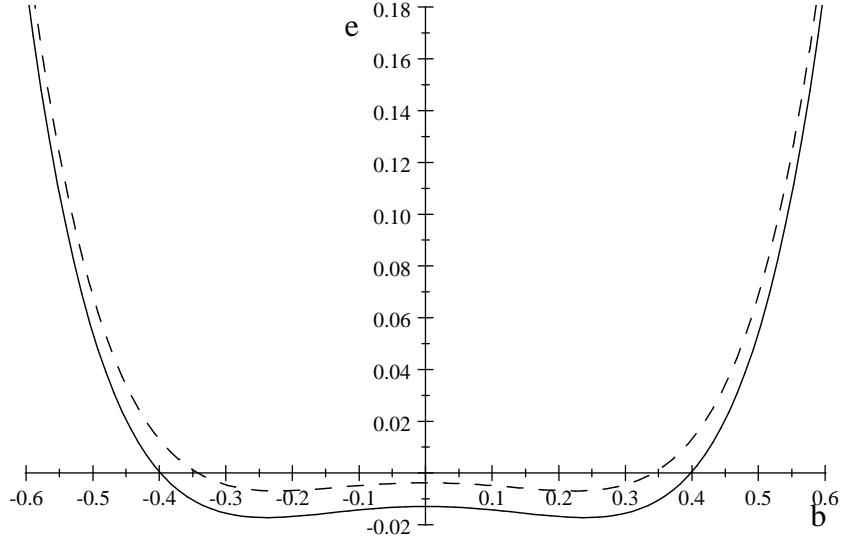


FIG. 9: The $2+1$ space-time dimensions effective potential as a function of the one-point function measured in units of i for $H = -1, G = +\frac{1}{2}$ (dashed) and $H = -1, G = -\frac{1}{2}$ (solid) with the stable solution $t = \frac{1}{180H} \left(180H - 60Hb^2 + 12G + 6\sqrt{40H^2b^4 + 4G^2 - 10H} \right)$ of the \mathcal{PT} -symmetric ϕ^6 potential .